# Using Quantifiers

Note Title 19/10/2011

This lecture uses a simple optimisation problem to illustrate the quantifier calculus. At the same time, we demonstrate the technique called *dynamic programming* that is often used to solve such problems.

0. Specify the problem (putative solutions, objective fn.)
1. Construct inductive formulation of putative solutions.
2. Use "principle of optimality" (in this case, addition distributes over max) to constructive inductive definition of optimal solution.
3. Use "topological search" to compute optimal solution.

# 0. Problem Specification

A set S of numbers is said to be *nc* the number N (written $S \text{ nc } N$) if

$$S \subseteq \{i \mid 0 \leq i < N\}$$

and

$$\langle \forall i : i \in S : i+1 \notin S \rangle .$$

Suppose $0 \leq N$ and A is an array of numbers of length N. Design an algorithm to calculate

$$\langle \Uparrow S : S \text{ nc } N : \langle \Sigma i : i \in S : A[i] \rangle \rangle$$

(nc is an abbreviation of non-consecutive).

$A: \; 4, 1, 5, 4, -2, 5, 3$

Example

$\emptyset$ nc 7

$\{2\}$ nc 7

$\{1,3,5\}$ nc 7

$\{1,4,6\}$ nc 7

$\neg(\{1,2\}$ nc $7)$

$\neg(\{1,3,4\}$ nc $7)$

| $S$ | $\langle \Sigma i : i \in S : A[i] \rangle$ |
|---|---|
| $\emptyset$ | 0 |
| $\{2\}$ | 5 |
| $\{1,3,5\}$ | 10 |
| $\{0,4,6\}$ | 5 |

# Notation

Abbreviate $S \subseteq \{i \mid 0 \leq i < N\}$ to $S$ sub $N$ .

Abbreviate $\langle \forall i : i \in S : i+1 \notin S \rangle$ to non-con.$S$ .

Then [ $S$ nc $N \equiv S$ sub $N \wedge$ non-con.$S$ ]

Abbreviate

$\langle \Uparrow S : S \text{ nc } N : \langle \Sigma i : i \in S : A[i] \rangle \rangle$

to opt.$N$ .

Note:

$$
\begin{array}{ll}
 & \langle \forall i : i \in S : i+1 \notin S \rangle \\
= & \quad \{ \text{ trading } \} \\
 & \langle \forall i :: i \in S \Rightarrow i+1 \notin S \rangle \\
= & \quad \{ \text{ implication } \} \\
 & \langle \forall i :: i \notin S \vee i+1 \notin S \rangle \\
= & \quad \{ \vee \text{ is symmetric, implication } \} \\
 & \langle \forall i :: i+1 \in S \Rightarrow i \notin S \rangle \\
= & \quad \{ \text{ trading } \} \\
 & \langle \forall i : i+1 \in S : i \notin S \rangle \ .
\end{array}
$$

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## Step 1: Seek an inductive definition of nc.

**Basis**

$$S \text{ nc } 0 \equiv S = \emptyset \ .$$

**Induction step**

$$
\begin{array}{ll}
 & S \text{ sub } k+1 \\
= & \quad \{ \text{ definition } \} \\
 & S \subseteq \{ i \mid 0 \leq i < k+1 \} \\
= & \quad \{ i < k+1 \equiv i < k \vee i = k \ \} \\
 & S \subseteq \{ i \mid 0 \leq i < k \} \vee \langle \exists T : S = T \cup \{k\} : T \subseteq \{ i \mid 0 \leq i < k \} \rangle \\
= & \quad \{ \text{ definition } \} \\
 & S \text{ sub } k \quad \vee \quad \langle \exists T : S = T \cup \{k\} : T \text{ sub } k \rangle
\end{array}
$$

## Induction step (continued)

Suppose $S$ sub $k$.

Then $\quad$ non-con.$(S \cup \{k\})$

$= \quad \{$ definition $\}$

$\langle \forall i : i \in S \cup \{k\} : i+1 \notin S \cup \{k\} \rangle$

$= \quad \{$ range splitting $\}$

$\langle \forall i : i \in S : i+1 \notin S \cup \{k\} \rangle$

$\wedge \langle \forall i : i \in \{k\} : i+1 \notin S \cup \{k\} \rangle$

$= \quad \{$ trading (see above); one-pt. rule, assumption: $S$ sub $k$ $\}$

$\langle \forall i : i+1 \in S \cup \{k\} : i \notin S \rangle$

$= \quad \{$ range splitting, one-pt. rule $\}$

$\langle \forall i : i+1 \in S : i \notin S \rangle \wedge k-1 \notin S$

$= \quad \{$ trading (see above), definition $\}$

non-con.$S \wedge k-1 \notin S$ .

I.e.

$[\, S$ sub $k \wedge$ non-con.$(S \cup \{k\}) \equiv S$ sub $k \wedge$ non-con.$S \wedge k-1 \notin S \,]$

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$S$ nc $k+1$

$= \quad \{$ definition of nc $\}$

$S$ sub $k+1 \wedge$ non-con.$S$

$= \quad \{$ induction step above $\}$

$(S$ sub $k \vee \langle \exists T : S = T \cup \{k\} : T$ sub $k \rangle)$

$\wedge$ non-con.$S$

$= \quad \{$ distributivity of $\wedge$ over $\vee$ $\}$

$(S$ sub $k \wedge$ non-con.$S)$

$\vee \langle \exists T : S = T \cup \{k\} : T$ sub $k \wedge$ non-con.$S \rangle$

$= \quad \{$ see below $\}$

$S$ nc $k \vee \langle \exists T : S = T \cup \{k\} : T$ nc $k-1 \rangle$

(Final step in above calculation.)

$$\langle \exists T :\ S = T \cup \{k\} :\ T \text{ sub } k \ \wedge\ \text{non-con}.S \rangle$$

$=$ { Leibniz }

$$\langle \exists T :\ S = T \cup \{k\} :\ T \text{ sub } k \ \wedge\ \text{non-con}.(T \cup \{k\}) \rangle$$

$=$ { separate calculation on page 7 }

$$\langle \exists T :\ S = T \cup \{k\}$$
$$:\ T \text{ sub } k \ \wedge\ \text{non-con}.T \ \wedge\ k-1 \notin T \rangle$$

$=$ { definition of sub and nc }

$$\langle \exists T :\ S = T \cup \{k\} :\ T \text{ nc } k-1 \rangle$$

## Summary (Inductive definition of putative solns.)

### Basis

$$S \text{ nc } 0 \ \equiv\ S = \emptyset .$$

$$S \text{ nc } -1 \ \equiv\ S = \emptyset .$$

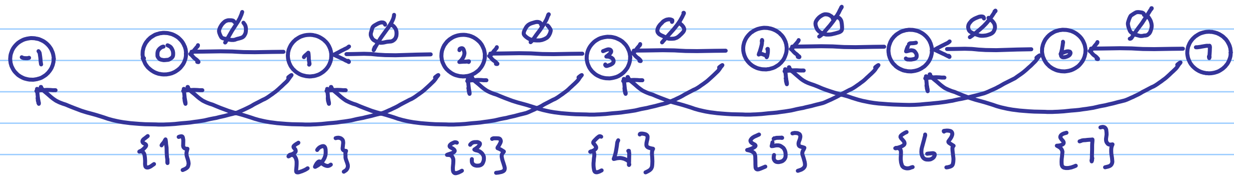
### Induction step

$$S \text{ nc } k+1$$

$=$

$$S \text{ nc } k \ \vee\ \langle \exists T :\ S = T \cup \{k\} :\ T \text{ nc } k-1 \rangle$$

# Graph Representation




$S \text{ nc } k{+}1$

$= \quad \langle \exists T : S = T \cup \emptyset : T \text{ nc } k \rangle \quad \vee \quad \langle \exists T : S = T \cup \{k\} : T \text{ nc } k{-}1 \rangle$

$[\ opt.N = \langle \Uparrow S : S \text{ nc } N : \langle \Sigma i : i \in S : A[i] \rangle \rangle\ ]$

**Basis**

$opt.0$

$=$ { definition }

$\langle \Uparrow S : S \text{ nc } 0 : \langle \Sigma i : i \in S : A[i] \rangle \rangle$

$=$

$\langle \Uparrow S : S = \emptyset : \langle \Sigma i : i \in S : A[i] \rangle \rangle$

$=$ { one-pt. rule }

$\langle \Sigma i : i \in \emptyset : A[i] \rangle \rangle$

$=$ { empty range }

$0$ .

Similarly, $opt.(-1) = 0$ .

## Induction step

$$
\begin{array}{ll}
 & opt.(k+1) \\
= & \{\ \text{definition}\ \} \\
 & \langle \Uparrow S : S \ nc \ k+1 \ : \langle \Sigma i : i \in S : A[i] \rangle \rangle \\
= & \{\ \text{inductive definition of } nc\ \} \\
 & \langle \Uparrow S : S \ nc \ k \ \vee \ \langle \exists T : S = T \cup \{k\} : T \ nc \ k-1 \rangle \\
 & \qquad : \langle \Sigma i : i \in S : A[i] \rangle \\
 & \rangle \\
= & \{\ \text{range disjunction and trading}\ \} \\
 & \qquad \langle \Uparrow S : S \ nc \ k \ : \langle \Sigma i : i \in S : A[i] \rangle \rangle \\
 & \uparrow \ \langle \Uparrow T : T \ nc \ k-1 : \\
 & \qquad\qquad \langle \Uparrow S : S = T \cup \{k\} : \langle \Sigma i : i \in S : A[i] \rangle \rangle \\
 & \qquad \rangle
\end{array}
$$

$$
\begin{array}{ll}
= & \{\ \text{definition of } opt, \ \text{one-pt. rule}\ \} \\
 & \quad opt.k \\
 & \uparrow \ \langle \Uparrow T : T \ nc \ k-1 : \langle \Sigma i : i \in T \cup \{k\} : A[i] \rangle \rangle \\
= & \{\ \text{range disjunction and one-pt. rule}\ \} \\
 & \quad opt.k \\
 & \uparrow \ \langle \Uparrow T : T \ nc \ k-1 : A[k] + \langle \Sigma i : i \in T : A[i] \rangle \rangle \\
= & \{\ \text{addition distributes through max}\ \} \\
 & \quad opt.k \\
 & \uparrow \ (A[k] + \langle \Uparrow T : T \ nc \ k-1 : \langle \Sigma i : i \in T : A[i] \rangle \rangle ) \\
= & \{\ \text{definition of } opt\ \} \\
 & \quad opt.k \ \uparrow \ (A[k] + opt.(k-1))
\end{array}
$$

# Summary (Inductive definition of opt)

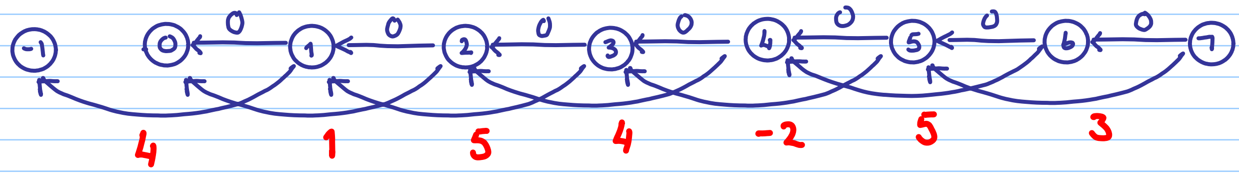
## Basis

$$\mathrm{opt}.0 = 0$$
$$\mathrm{opt}.(-1) = 0$$

## Induction step

$$\mathrm{opt}.(k+1) = \mathrm{opt}.k \uparrow (A[k] + \mathrm{opt}.(k-1))$$

A: 4, 1, 5, 4, -2, 5, 3




Compute longest path.

$\{0 \leq N\}$

$k, m, n \;:=\; 0, 0, 0\;;$

$\{$ Invariant : $\quad m = opt.k \;\wedge\; n = opt.(k-1)$

**do** $\quad k \neq N \;\rightarrow\; m, n, k \;:=\; m \uparrow (A[k]+n),\, m,\, k+1$

**od**

$\{\; m = opt.N \;\}$

Count no. of putative solutions.

$\langle \Sigma S :\; S \text{ nc } N :\; 1 \rangle$